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Propagation of shape-preserving optical pulses in inhomogeneously broadened multi-level systems

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Abstract. We study stable propagation of multiple shape-preserving optical pulses in an inhomogeneously broadened multi-level atomic medium. By analytically solving the Maxwell-Schrödinger equations governing the evolution of N coupled optical fields and atomic amplitudes we show that N pulsed optical waves coupling to $(N+1)$-levels can be automatically matched with the same soliton waveform and identical yet very slow propagation velocity. Several sets of coupled soliton solutions for two different $(N+1)$-level models are given and their stability is studied by using a numerical simulation.

PACS. 42.50.Md Optical transient phenomena: quantum beats, photon echo, free-induction decay, dephasings and revivals, optical nutation, and self-induced transparency – 42.50.Gy Effects of atomic coherence on propagation, absorption, and amplification of light; electromagnetically induced transparency and absorption

1 Introduction

Solitons are intriguing nonlinear localization phenomena occurring in many branches of physics. Solitons have been discovered in many states of matter ranging from solid medium, such as optical fiber (optical soliton [1,2]), to Bose-Einstein condensed atomic vapor (matter wave solitons [3,4]). In recent years, there has been intensive interest in the study of solitons in resonant optical media, including the self-induced transparency (SIT) [5,6] in two-level atoms, normal-mode [7] and optical simultons in three- [8–11], four- [12], and five-level [9] media, lasing without inversion [13], phaseonium [14], electromagnetically induced transparency (EIT) [15], and ultra slow optical solitons [16,17]. A constant focus of interest is to obtain analytical coupled soliton solutions of Maxwell-Schrödinger (MS) equations governing the evolution of optical pulses and atomic-state probability amplitudes.

Recently, much attention has been paid to the pulse propagation in coherently prepared multi-level systems [18–22,24,25]. It has been shown that a multi-level system may posses multiple dark states, and therefore can be used to realize many interesting quantum interference effects including coherent population trapping and transfer, multiple channel EIT processes, and dynamic group velocity control. Many theoretical approaches for solving the coupled MS equations of multi-level systems involve linearization method in conjunction with steady-state or adiabatic approximations [18–22,24,25]. However, such techniques exclude the possibility of shape-preserving soliton solutions which are resulted from inherently nonlinear processes.

In this work, we investigate stable propagation of N (N may be larger than 4) optical pulses in $(N+1)$-level atomic systems. We go beyond the linearization and steady state approximations and show analytically that a stable propagation of N well-matched optical pulses with a form of coupled optical solitons is possible. These analytical results for nonlinear propagation in multi-level systems, to the best of our knowledge, are not available in the literature. We also show that similar to the case of SIT of two-level atoms, exact coupled soliton solutions still exist even when inhomogeneous broadening such as Doppler effect is taken into account. The propagation velocity of the coupled optical solitons may be controlled by manipulating the intensity of pump fields and atomic density, and can be made much less than the speed of light in vacuum. We further show that the N coupled optical solitons, within our $(N+1)$-level models, can be well matched, resulting in N matched and coupled solitons of the same waveform and group velocity. The paper is arranged as follows. In Section 2 we describe two different $(N+1)$-level models and present the equations of motion governing the evolution of atomic amplitudes and laser fields. In Sections 3 and 4 we provide analytical solutions of N coupled solitons for a generalized Λ-type and a V-type system subject to

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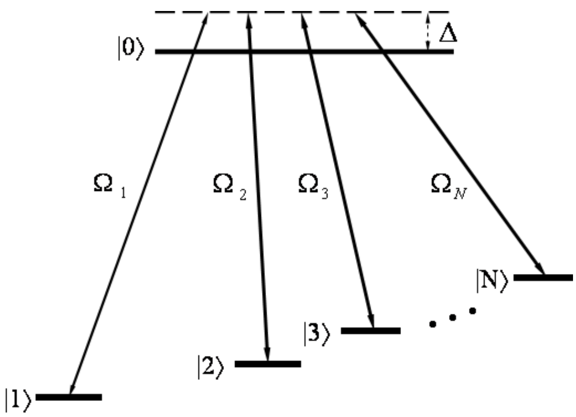

Fig. 1. Schematic energy-level diagram of the generalized Λ-type system, in which an upper-level is coupled to N lower-levels by N nearly resonant laser fields.

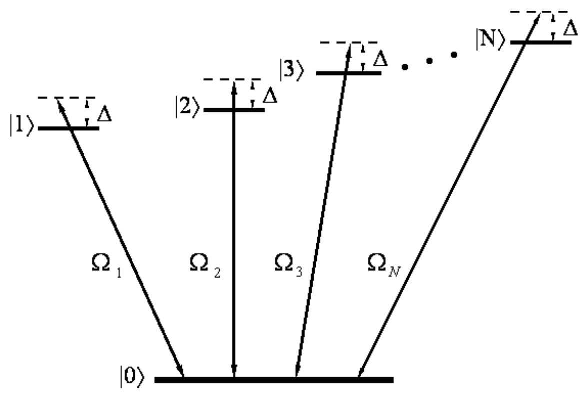

Fig. 2. Schematic energy-level diagram of the generalized V-type system, in which a lower-level is coupled to N upper-levels by N nearly resonant laser fields.

inhomogeneous broadening, respectively. In particular, we show analytically the possibility of superluminal N coupled and matched optical solitons in a V-type $(N+1)$-level system. We note that there has been no report in literature on multiple well-matched and coupled superluminal optical solitons. In addition, we also discuss pulse propagation and population transfer described by the coupled soliton solutions. In Section 5, we perform a numerical simulation to study the stability of the coupled soliton solutions. The last section (Sect. 6) contains discussion and conclusion of our results.

2 Models and nonlinear evolution equations

We consider the propagation of N optical pulses that are simultaneously propagating in z-direction and are interacting resonantly with an emsamble of $(N+1)$-level atomic systems. For simplicity we neglect the inhomogeneity of the medium. The electric-field vector for the N optical pulses can be written as

$$\mathbf{E} = \sum_{j=1}^{N} \mathbf{e}_j \mathcal{E}_j(z,t) \exp[i(k_j z - \omega_j t)] + \text{c.c.}, \tag{1}$$

where k_j, ω_j, $\mathcal{E}_j$ and $\mathbf{e}_j$ are the wavevector, frequency, amplitude and polarization direction of the jth optical pulse, respectively. We consider two different $(N+1)$-level models. One is a generalized multi-channel Λ-type system where N laser pulses couple N lower levels $|j\rangle$ $(j = 1, 2, ..., N)$ to a single excited upper-level $|0\rangle$ (see Fig. 1). Such a model, here called N-pod system, was first mentioned by Morris and Shore [23] when studying coherent excitations in multi-state systems. Since these lower levels have different energy and hence, generally, N optical pulses with different frequency should be applied in order to excite all possible transitions in the system. The N lower levels in such system can be obtained from Zeeman or hyperfine split of atomic ground state. The N-pod system is a natural and direct generalization of a tripod system [22] and has been widely studied recently in relation to multiple dark states, coherent population transfer and multi-channel EIT. For pulse propagation in tri- and N-pod systems, the reader is referred to references [18–22,24,25]. The other model we study is a generalized multi-channel V-type system, where N laser pulses couple N upper-levels $|j\rangle$ $(j = 1, 2, ..., N)$ to a single ground level $|0\rangle$ (see Fig. 2) [26].

The Hamiltonian of both systems has the form $\hat{H} = \hat{H}_0 + \hat{H}'$, where $\hat{H}_0$ describes an free atom and $\hat{H}'$ describes the interaction between the atom and the optical field. In Schrödinger picture, the state vector of the systems is expressed by $|\Psi(t)\rangle = \sum_{j=0}^{N} c_j(z,t)|j\rangle$, where $|j\rangle$ is the eigenstate of $\hat{H}_0$. Under electric-dipole and rotating-wave approximations [27], the Hamiltonian takes the form

$$\hat{H} = \sum_{j=0}^{N} \epsilon_j |j\rangle\langle j| - \hbar \left[\sum_{j=1}^{N} \Omega_j(z,t) \times \exp[i(k_j z - \omega_j t)]|0\rangle\langle j| + \text{H.c.} \right], \ (\Lambda-\text{type}) \tag{2a}$$

$$\hat{H} = \sum_{j=0}^{N} \epsilon_j |j\rangle\langle j| - \hbar \left[\sum_{j=1}^{N} \Omega_j(z,t) \times \exp[i(k_j z - \omega_j t)]|j\rangle\langle 0| + \text{H.c.} \right], \ (V-\text{type}) \tag{2b}$$

where ϵ_j is the energy of state $|j\rangle$, $\Omega_j = \mathbf{e}_j \cdot \mathbf{p}_{0j} \mathcal{E}_j/\hbar$ $(\Omega_j = \mathbf{e}_j \cdot \mathbf{p}_{j0} \mathcal{E}_j/\hbar)$ is the half Rabi frequency corresponding to jth optical pulse for the Λ-type (V-type) system, $\mathbf{p}_{0j}$ is the electric dipole matrix element associated with the transition from $|0\rangle$ to $|j\rangle$, and H.c. represents Hermitian conjugate.

To investigate the time evolution of the system it is more convenient to employ an interaction picture, which is obtained by making the transformation $c_j(z,t) = a_j(z,t) \exp\{i[(k_1 - k_j)z - (\epsilon_j/\hbar + \Delta_j)t)]\}$ and $c_j(z,t) = a_j(z,t) \exp\{i[k_j z - (\epsilon_j/\hbar + \Delta_j)t]\}$ (with $k_0 = 0$) for the Λ-type and V-type system, respectively. The Hamiltonian in the interaction picture reads

$$\hat{H}_{\rm int} = -\hbar \sum_{j=0}^{N} \Delta_j |j\rangle\langle j| - \hbar \left[\sum_{j=1}^{N} \Omega_j(z,t) |0\rangle\langle j| + {\rm H.c.} \right], \quad (\Lambda-\text{type}) \quad (3a)$$

$$\hat{H}_{\rm int} = -\hbar \sum_{j=0}^{N} \Delta_j |j\rangle\langle j| - \hbar \left[\sum_{j=1}^{N} \Omega_j(z,t) |j\rangle\langle 0| + {\rm H.c.} \right], \quad (V-\text{type}). \quad (3b)$$

Using the Schrödinger equation $i\hbar\partial|\Psi(t)\rangle_{\rm int}/\partial t = H_{\rm int}|\Psi(t)\rangle_{\rm int}$ and the Maxwell equation $\nabla^2\mathbf{E} - (1/c^2)\partial^2\mathbf{E}/\partial t^2 = [1/(\epsilon_0 c^2)]\partial^2\mathbf{P}/\partial t^2$[28] one can readily obtain the MS equations describing the evolution of a_j and Ω_j $(j = 1, 2, ..., N)$

$$i\left(\frac{\partial}{\partial t} + \Delta_0\right) a_0 + \sum_{l=1}^{N} R_l a_l = 0, \quad (4a)$$

$$i\left(\frac{\partial}{\partial t} + \Delta_j\right) a_j + R_j^* a_0 = 0, \quad (4b)$$

$$i\left(\frac{\partial}{\partial z} + \frac{1}{c}\frac{\partial}{\partial t}\right) \Omega_j + \kappa_0 \int d\Delta g(\Delta)\, P_0 P_j^* = 0, \quad (4c)$$

with the normalization condition $\sum_{l=0}^{N} |a_l|^2 = 1$. In equation (4c), $g(\Delta)$ is the distribution function of detuning resulted from an inhomogeneous (Doppler-type) broadening. Propagation coefficients are defined by $\kappa_{0j} = \mathcal{N}_a \omega_j |\mathbf{p}_{j0}|/(2\epsilon_0 c\hbar)$ with $\mathcal{N}_a$ being atomic density and c being the light speed in vacuum. For simplicity all propagation coefficients are assumed to be equal [10,11] (i.e., $\kappa_{0j} = \kappa_0$). Experimentally, this can be achieved using an atomic element with ground (excited) state having large angular momenta J or F. Parameters Δ_0 and Δ_j are related to the detuning Δ and functions R_j, P_0 and P_j are related to the atomic amplitudes a_0 and a_j, and Rabi frequency Ω_j. Table 1 shows the correspondence of these symbols and quantities for the generalized Λ-type and the V-type systems. Since we are interested in the pulse propagation in coherent transient regime, the damping in equation (4) due to the finite lifetime of the excited-state levels have been neglected. Such approximation is valid for pulses having the temporal width short enough so that relaxation terms related to homogeneous broadening in the MS equations take no significant role.

Equations (4a–4c) are nonlinearly coupled wave equations with dispersion and inhomogeneous broadening effects included. In general, a laser field tuned close to an atomic line or on the wing of the Doppler broadened profile will experience increased loss and distortion as it is tuned closer to the line center [29]. However, under suitable conditions such losses and distortions can be compensated by certain nonlinearity of the system, resulting in a lossless soliton-like wave propagation, as will be shown below.

Table 1. Physical quantities used in equation (4) for different types of medium.

Symbols in Eq. (4)	Λ-type system	V-type system
R_j	Ω_j	Ω_j^*
P_0	a_0	a_0^*
P_j	a_j	a_j^*
Δ_0	Δ	0
Δ_j	0	Δ

3 Coupled solitons for a generalized multi-channel Λ-type system

Since in general there is no known method to solve the coupled equations (4a–4c), we turn to find some of their particular solutions using the standard trial solution method [11]. For the coupled soliton solution of a generalized Λ-type system shown in Figure 1 we assume that $a_j = a_j(\zeta)$ and $\Omega_j = \Omega_j(\zeta)$ with $\zeta = Kz - \tau/\tau_0$. Here $\tau = t - z/c$ and K and τ_0 are two real parameters to be determined.

We first assume the following trial solutions: $a_0 = iA_0 {\rm sech}\zeta \exp[i\phi(z)]$, $a_1 = A_{10} + A_{11}\tanh\zeta$, $a_j = A_j {\rm sech}\zeta \exp[i\phi(z)]$, $\Omega_1 = B_1 {\rm sech}\zeta \exp[i\phi(z)]$, and $\Omega_j = B_j \tanh\zeta$ $(j = 2, 3, ..., N)$ [30]. Physically, these trial solutions describe matched propagations (in the sense of the same group velocity) of one bright soliton and $N-1$ dark solitons, and the corresponding evolution of the atomic population distribution. In the atom rest frame, an atom with all population initially in the ground state $|1\rangle$ will first see the arrival of $N-1$ strong fields Ω_j (corresponding to the $\zeta = -\infty$ limit). As the intensity of the $N-1$ fields decrease, indicating the arrival of matched $N-1$ dark solitons, the atom also sees the increase of field Ω_1, signaling the arrival of the bright soliton which reaches the peak intensity at $\zeta = 0$ where the intensities of the dark solitons become zero. Beyond $\zeta = 0$, the atom sees the decrease of the intensity of Ω_1 and the increase of the intensities of Ω_j, indicating the passing of the bright and dark solitons. During the transient period of bright and dark solitons, the population of the atom changes from all in the ground state $|1\rangle$, to shared by lower states, and then back to the ground state $|1\rangle$. In regions where $|\zeta| > 0$ and $|\Omega_j| > |\Omega_1|$, the system is generally in a conventional single channel EIT regime where the fields Ω_j drive the upper state $|2\rangle$ transparent for the weaker field Ω_1. In the region where $|\zeta| \geq 0$ but $|\Omega_j| < |\Omega_1|$ the conventional EIT can still become operative with the role of field transparency switched. It is in this region the $N-1$ transparency channels may be established by a single driving field Ω_1 (see Fig. 3).

Substituting the above trial solutions into equation (4), we get a set of nonlinear algebraic equations for the parameters A_0, A_{10}, A_{11}, A_j and B_j (see Appendix A). Solving these algebraic equations we obtain $A_0 = B_1\tau_0/(1-i\Delta\tau_0)$, $A_{10} = i\Delta\tau_0/(1-i\Delta\tau_0)$, $A_{11} = 1/(1-i\Delta\tau_0)$, $A_j = -B_j/[B_1(1-i\Delta\tau_0)]$ $(j = 2, 3, ..., N)$. Thus, we have the following N coupled and group velocity matched soliton

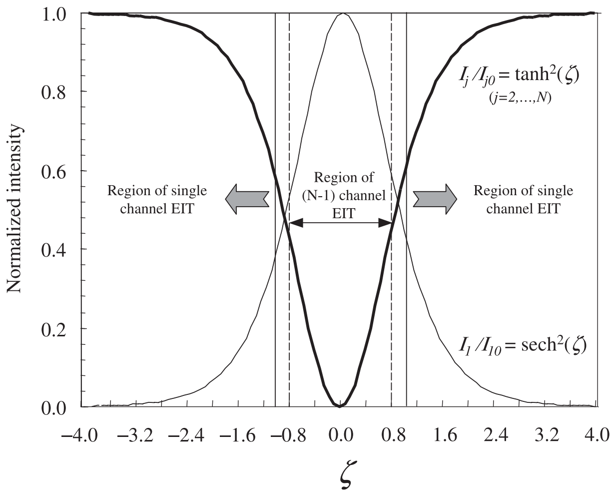

Fig. 3. Schematic drawing of intensity distributions for the N fields and the regions corresponding to single channel and $(N-1)$ channels EIT that may be achieved with a generalized Λ-type system that admits N coupled optical solitons. Atoms are assumed to be initially in the state $|1\rangle$. Similar plots can be made for other sets of solutions described in the the text.

solution:

$$a_0 = \frac{i\tau_0 B_1}{1-i\Delta\tau_0}\mathrm{sech}\zeta\, e^{i\phi(z)}, \tag{5a}$$

$$a_1 = \frac{1}{1-i\Delta\tau_0}\left(i\Delta\tau_0 + \tanh\zeta\right), \tag{5b}$$

$$a_j = -\frac{B_j}{B_1(1-i\Delta\tau_0)}\mathrm{sech}\zeta\, e^{i\phi(z)}, \tag{5c}$$

$$\Omega_1 = B_1\mathrm{sech}\zeta\, e^{i\phi(z)}, \tag{5d}$$

$$\Omega_j = B_j\tanh\zeta, \quad (j=2,3,...,N) \tag{5e}$$

where $\tau_0 = (B_1^2 - \sum_{l=2}^N B_l^2)^{1/2}/B_1^2$, and

$$K = \frac{\kappa_0}{\tau_0}\int d\Delta\frac{g(\Delta)}{\Delta^2+(1/\tau_0)^2}, \tag{6a}$$

$$\frac{d\phi}{dz} = \kappa_0\int d\Delta\frac{\Delta g(\Delta)}{\Delta^2+(1/\tau_0)^2}. \tag{6b}$$

Here B_j $(j=1,2,...,N)$ are N arbitrary constants subject to the constraint $B_1^2 \geq \sum_{l=2}^N B_l^2$, i.e., the peak intensity of the field Ω_1 must be greater than the peak intensities of all $N-1$ fields Ω_j $(j=2,3,...,N)$ combined.

With the assumption of homogeneously distributed medium, K is a z-independent real constant. Thus, from equation (6b) one gets $\phi(z) = K'z$ with $K' = \kappa_0\int d\Delta g(\Delta)\Delta/[\Delta^2+(1/\tau_0)^2]$ being a constant [31]. This indicates that for an $(N+1)$-level system even in the presence of inhomogeneous broadening, non-distorted propagation of shape-preserving optical pulses of soliton type is indeed possible. The inhomogeneous broadening, however, will affect the propagation velocity of the N coupled solitons through the constant K. The matched group velocity V for the N coupled solitons in the presence of the inhomogeneous broadening is determined by

$$\frac{1}{V} = \frac{1}{c} + \frac{\kappa_0}{B_1^2\tau_0^2}\int d\Delta\frac{g(\Delta)}{\Delta^2+(1/\tau_0)^2}. \tag{7}$$

With sufficiently high atomic number density, the second term can dominate the group velocity even with modest field intensity, i.e., B_1^2. When this occurs the propagation velocity V becomes much less the speed of light in vacuum and one obtains slow propagation of N coupled optical solitons. To demonstrate this effect, we assume the medium has a Gaussian line-shape function [11]

$$g(\Delta) = \frac{T_2^*}{\sqrt{2\pi}}\exp\left\{-\frac{[(\Delta-\Delta_0)T_2^*]^2}{2}\right\}, \tag{8}$$

where Δ_0 is the detuning from line center and T_2^* is inhomogeneous lifetime. For simplicity, we assume $\Delta_0 = 0$ and hence $\int d\Delta g(\Delta)/(\Delta^2 + 1/\tau_0^2) = \sqrt{\pi/2}\,T_2^*\tau_0\exp[T_2^{*2}/(2\tau_0^2)]$. If we take $T_2^* = \tau_0$, $\kappa_0 = 1.0\times10^{14}$ cm^{-1} s^{-1} and $B_1 = 2.0\times10^{10}$ s^{-1}, using equation (7) we get $V = 1.75\times10^{-4}c$. This is a very slow propagation velocity. Thus, the solution equation (5) describe N coupled slow-optical solitons [32] in an $(N+1)$-level system. The stable propagation of such N coupled optical solitons in an inhomogeneously broadened $(N+1)$-level system is the result of exact balance between nonlinearity and dispersion of the system [33]. Obviously, the two-coupled-soliton solution of a three-level Λ-type system obtained by Rahman and Eberly [11] is a special case of our result (Eq. (5)) for $N=2$.

Using a similar procedure we can also get a different type of N coupled and matched soliton solution of a generalized Λ-type system:

$$a_0 = \frac{i}{1-i\Delta\tau_0}\mathrm{sech}\zeta\, e^{i\phi(z)}, \tag{9a}$$

$$a_j = \frac{B_j\tau_0}{1-i\Delta\tau_0}\left(-i\Delta\tau_0 + \tanh\zeta\right), \tag{9b}$$

$$\Omega_j = B_j\,\mathrm{sech}\zeta\, e^{i\phi(z)}, \quad (j=1,2,...,N), \tag{9c}$$

where $\tau_0 = (\sum_{l=1}^N B_l^2)^{-1/2}$. The expressions of K and $d\phi/dz$ are still given by equations (6a) and (6b).

Equations (9a–9c) describe a process that all light fields are not established initially (i.e., at $\zeta=-\infty$) and particles are populated in N lower-levels. An experimental correspondence of this initial condition is an atomic system that has a ground state with large total angular momenta, providing a large Zeeman sub-level family in the presence of a magnetic field. The population probability in the jth level is $n_j = B_j^2/\sum_{l=1}^N B_l^2$. At $\zeta=0$ all light fields have reached maximum intensity and all particles are in an N-states superposition state. At $\zeta=\infty$, all particles return back into the N lower levels. This final system state, however, can be fundamentally different from the initial system state even though it appeared that the population is shared by all N lower levels. The key difference is that if all N lower states belong to the same hyperfine manifold with long coherence time, one essentially has a phase-correlated medium as the result of coherent population transfer [14]. We note that the N coupled optical solitons given in equation (9) are well matched, i.e., they have identical temporal-spatial intensity profile and group velocity [34]. The matched group velocity is given by

$$\frac{1}{V} = \frac{1}{c} + \kappa_0 \int d\Delta \frac{g(\Delta)}{\Delta^2 + (1/\tau_0)^2}. \tag{10}$$

Note again that this group velocity can be made very small using large atomic density $\mathcal{N}_a$ and small optical pulse amplitudes B_j. Equation (9) demonstrates N *well-matched optical solitons* in an $(N+1)$-level Λ-type of system. To the best of our knowledge, no such results are available in literature.

4 Coupled solitons for a multi-channel V-type system

We now consider a generalized V-type system, shown in Figure 2, and seek exact solutions of equations (4a–4c) (see Tab. 1 for parameters and quantities). We assume trial solutions $a_0 = i(A_{00} + A_{01}\tanh\zeta)$, $a_j = A_j \mathrm{sech}\zeta\, e^{i\phi(z)}$, $\Omega_j = B_j \mathrm{sech}\zeta\, e^{i\phi(z)}$ $(j = 1, 2, ..., N)$. The physical consideration to get these trial solutions is that initially (i.e., $\zeta = -\infty$), the population is in the ground state $|0\rangle$ and no field is present. At $\zeta = 0$, all light fields have reached their maximum intensities and significant fraction (theoretically speaking, 100%) of population has been transferred and disbursed from the ground state $|0\rangle$ to the excited state $|j\rangle$ $(j = 1, 2, ..., N)$. At $\zeta = \infty$, particles return back into the ground state $|0\rangle$.

Substituting the above trial solutions into equations (4a–4c) we obtain a set of algebraic equations for the parameters A_{00}, A_{01}, A_j and B_j. Solving these algebraic equations we obtain $A_{00} = -i\Delta\tau_0/(1 - i\Delta\tau_0)$, $A_{01} = 1/(1 - i\Delta\tau_0)$, $A_j = -B_j\tau_0/(1 - i\Delta\tau_0)$. Thus, we have the following N coupled and matched solitons solution:

$$a_0 = \frac{1}{1 - i\Delta\tau_0}\left(-i\Delta\tau_0 + \tanh\zeta\right), \tag{11a}$$

$$a_j = -\frac{B_j\tau_0}{1 - i\Delta\tau_0}\mathrm{sech}\zeta\, e^{i\phi(z)}, \tag{11b}$$

$$\Omega_j = B_j\,\mathrm{sech}\zeta\, e^{i\phi(z)}, \quad (j = 1, 2, ..., N), \tag{11c}$$

where $\tau_0 = \left(\sum_{l=1}^{N} B_l^2\right)^{-1/2}$ and B_j are arbitrary constants. The expression of K takes the same form as equation (6a) but now one has

$$\frac{d\phi}{dz} = -\kappa_0 \int d\Delta \frac{\Delta g(\Delta)}{\Delta^2 + (1/\tau_0)^2}. \tag{12}$$

Note that during propagation the N optical pulses are well matched. We further note that the coupled two-solitons-solution of a three-level V-type system [11] is a special case of equation (11) for $N = 2$.

To find some additional N coupled solutions solutions for the generalized V-type system we make the following transformation for atomic amplitudes:

$$a_0 = i\bar{b}_0(z,\tau)e^{i\Delta\tau}, \tag{13a}$$

$$a_j = \bar{a}_j(z,\tau)e^{i\Delta\tau}. \tag{13b}$$

Then the MS equations become

$$-\frac{\partial\bar{b}_0}{\partial\tau} - i\Delta\bar{b}_0 + \sum_{l=1}^{N}\bar{a}_l\Omega_l^* = 0, \tag{14a}$$

$$\frac{\partial\bar{a}_j}{\partial\tau} + \Omega_j\bar{b}_0 = 0, \tag{14b}$$

$$|\bar{b}_0|^2 + \sum_{l=1}^{N}|\bar{a}_l|^2 = 1, \tag{14c}$$

$$\frac{\partial\Omega_j}{\partial z} - \kappa_0\int d\Delta g(\Delta)\,\bar{a}_j\bar{b}_0^* = 0. \tag{14d}$$

To solve equations (14a–14d) we take the following trial solutions $\bar{b}_0 = A_0\mathrm{sech}\zeta\exp[i\phi(z)]$, $\bar{a}_1 = A_{10} + A_{11}\tanh\zeta$, $\Omega_1 = B_1\mathrm{sech}\zeta\exp[-i\phi(z)]$, $\bar{a}_j = A_j\mathrm{sech}\zeta\exp[i\phi(z)]$, and $\Omega_j = B_j\tanh\zeta$ $(j = 2, 3, ..., N)$. Substituting these trial solutions into equations (14a–14d) we get $A_0 = B_1\tau_0/(1 + i\Delta\tau_0)$, $A_{10} = i\Delta\tau_0/(1 + i\Delta\tau_0)$, $A_{11} = 1/(1 + i\Delta\tau_0)$, and $A_j = -B_j/[B_1(1 + i\Delta\tau_0)]$ $(j = 2, 3, ..., N)$. We then obtain a different set of N coupled and group velocity matched solitons solution:

$$a_0 = i\frac{B_1\tau_0}{1 + i\Delta\tau_0}\,\mathrm{sech}\zeta\, e^{i[\phi(z) + \Delta\tau]}, \tag{15a}$$

$$a_1 = \frac{1}{1 + i\Delta\tau_0}\left(i\Delta\tau_0 + \tanh\zeta\right)\, e^{i\Delta\tau}, \tag{15b}$$

$$a_j = -\frac{B_j}{B_1(1 + i\Delta\tau_0)}\,\mathrm{sech}\zeta\, e^{i[\phi(z) + \Delta\tau]}, \tag{15c}$$

$$\Omega_1 = B_1\,\mathrm{sech}\zeta\, e^{-i\phi(z)}, \tag{15d}$$

$$\Omega_j = B_j\,\tanh\zeta, \quad (j = 2, 3, ..., N) \tag{15e}$$

where $\tau_0 = B_1^{-1}\left(1 - \sum_{l=2}^{N} B_l^2/B_1^2\right)^{1/2}$, and

$$K = -\frac{\kappa_0}{\tau_0}\int d\Delta \frac{g(\Delta)}{\Delta^2 + (1/\tau_0)^2}. \tag{16}$$

The expression of $d\phi/dz$ is still given by equation (12). The common propagation velocity of the N coupled and group velocity matched solitons (15) is given by

$$\frac{1}{V} = \frac{1}{c} - \frac{\kappa_0}{B_1^2\tau_0^2}\int d\Delta \frac{g(\Delta)}{\Delta^2 + (1/\tau_0)^2}. \tag{17}$$

Notice the negative sign in equation (17). This is the consequence of initial condition. From equations (13a–13c) it is seen that initially, i.e., when $\zeta = -\infty$, the population is in the state $|1\rangle$. This corresponds to an inverted population and will lead to gain that is responsible for stimulated process and waveform reshaping. This waveform reshaping leads to apparent superluminal propagation, reflected

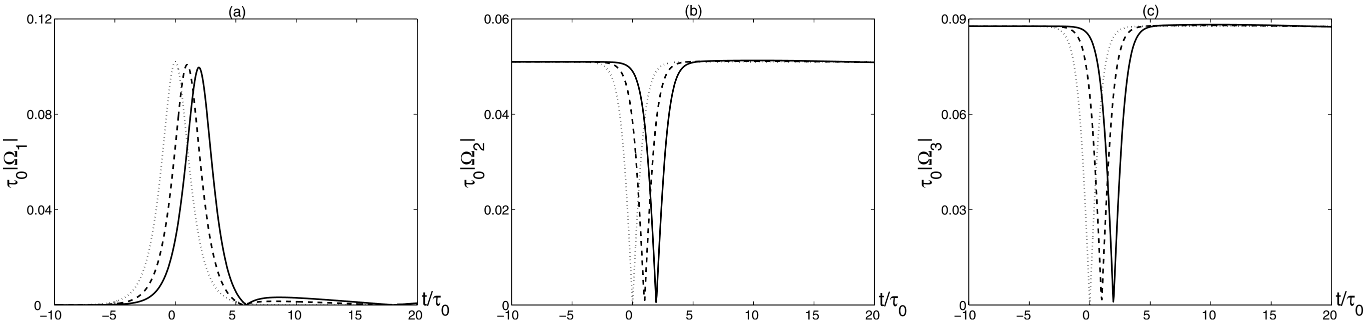

Fig. 4. The propagation of $\tau_0|\Omega_1|$ (panel (a)), $\tau_0|\Omega_2|$ (panel (b)) and $\tau_0|\Omega_3|$ (panel (c)) when taking the matched soliton solution (5) as an initial condition. The dot, dash and solid lines correspond to the propagating distance $z = 0$, 1.5 and 3.0 cm, respectively.

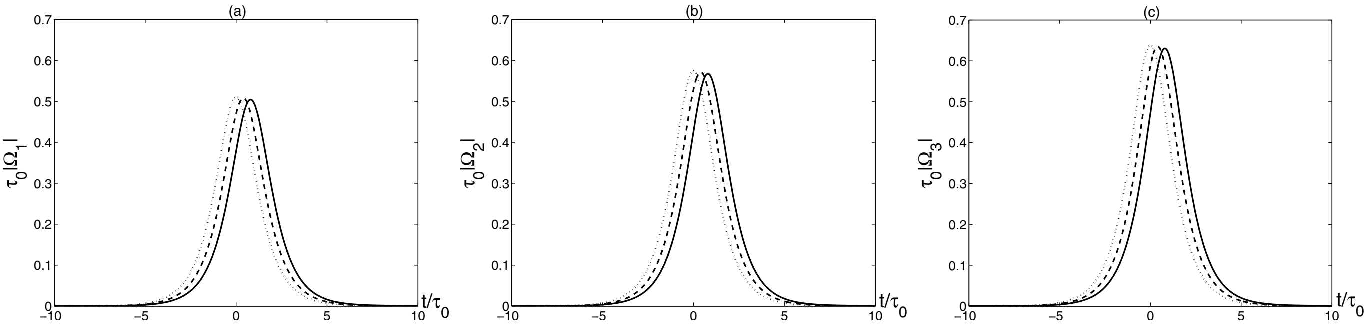

Fig. 5. The propagation of $\tau_0|\Omega_1|$ (panel (a)), $\tau_0|\Omega_2|$ (panel (b)) and $\tau_0|\Omega_3|$ (panel (c)) when taking the matched soliton solution (11) as an initial condition. The dot, dash and solid lines correspond to the propagating distance $z = 0.0$, 3.0 and 6.0 cm in each panel, respectively.

by the negative sign in equation (15). Indeed, with sufficiently high atomic density the second term can dominate even for modest intensity of the field Ω_1. Consequently, V can be negative and hence one obtains a *stable superluminal propagation of N coupled and group velocity matched optical optical solitons in a $(N+1)$-level V-type medium.*

With transformation (13b), we can get yet another set of N coupled and well-matched soliton solutions that satisfy equations (14a–14d). Indeed, it is readily shown that

$$a_0 = \frac{i}{1 + i\Delta\tau_0}\mathrm{sech}\zeta\, e^{i[\phi(z)+\Delta\tau]}, \tag{18a}$$

$$a_j = \frac{B_j\tau_0}{1 + i\Delta\tau_0}\,(i\Delta\tau_0 + \tanh\zeta)\; e^{i\Delta\tau}, \tag{18b}$$

$$\Omega_j = B_j\,\mathrm{sech}\zeta\, e^{i\phi(z)}, \quad (j = 1, 2, ..., N) \tag{18c}$$

is the exact solution of equations (14a–14d), where $\tau_0 = (\sum_{l=1}^{N} B_l^2)^{-1/2}$ and K and $d\phi/dz$ have the same form given by (6a) and (6b), respectively. With the similar technique it will not be surprised that other type of analytical solutions describing N coupled solitons can be found.

5 Numerical simulations

We now study, by using a numerical simulation, the stability of the coupled soliton solutions provided in the last two sections. In our computation, equations (4a–4c) are solved by using a fourth-order Runge-Kutta method (for Eqs. (4a) and (4b)) and a finite difference method (for Eq. (4c)). To test the stability of the soliton solutions, in the simulation the analytical results given above are naturally taken as initial conditions of equations (4a–4c).

Shown Figure 4 is the simulating result for the optical soliton propagation when taking the matched soliton solution (5) as an initial condition. Here, for illustration we select $N = 3$ (i.e. we simulate a 4-level Λ-type system) and the parameters of the system are chosen as $\kappa_0 = 1.0\times10^8$ cm^{-1} s^{-1}, $\Delta = 2.0\times10^8$ s^{-1}, $B_1 = 2.0\times10^9$ s^{-1}, $B_2 = 1.0\times10^9$ s^{-1} and $B_3 = 1.7\times10^9$ s^{-1}. Thus, in this case one has $\tau_0 \approx 8.3\times10^{-9}$ s. The evolution of the dimensionless Rabi frequencies $\tau_0|\Omega_1|$, $\tau_0|\Omega_2|$ and $\tau_0|\Omega_3|$ has been plotted in panel (a) to panel (c), respectively. In each panel, the dot, dash and solid lines correspond respectively to $z = 0$, 1.5 and 3.0 cm. From panel (a) we see that the bright soliton component (i.e. $\tau_0|\Omega_1|$) undergoes a slight deformation during propagation. At a large propagating distance some radiations (i.e. the waves with a very small amplitude) appear on the tail of the bright soliton. However, the dark soliton components (i.e. $\tau_0|\Omega_2|$ and $\tau_0|\Omega_3|$) show a very stable propagation.

Shown in Figure 5 is the simulating result for the propagation of optical pulses when the coupled optical soliton solution (11) is considered as an initial condition. In this case all Rabi frequencies $\tau_0|\Omega_1|$, $\tau_0|\Omega_2|$ and $\tau_0|\Omega_3|$ are bright solitons. In the simulation, we have taken $B_1 = 1.6\times10^9$ s^{-1}, $B_2 = 1.8\times10^9$ s^{-1}, $B_3 = 2.0\times10^9$ s^{-1}

and hence $\tau_0 \approx 3.2 \times 10^{-8}$ s. Other parameters are chosen the same as those used in Figure 4. In each panel we have plotted the waveform of $\tau_0|\Omega_1|$ (panel (a)), $\tau_0|\Omega_2|$ (panel (b)), and $\tau_0|\Omega_3|$ (panel (c)) at $z = 0, 3.0$ and 6.0 cm, respectively. We see that all bright solitons exhibit great robustness during their propagation even at a quite long distance.

6 Discussion and summary

In conclusion, we have investigated the stable propagation of shape-preserving optical pulses in two inhomogeneously broadened multi-level media. Using a trial solution method we have solved analytically the coupled Maxwell-Schrödinger equations governing the evolution of optical fields and atomic amplitudes. We have given analytical expressions of several sets of N coupled and well matched optical soliton solutions and tested their stability by using a numerical simulation. To the best of our knowledge, there has been no study in literature that reports N ($N > 4$) coupled optical solitons. In addition, we have shown that with suitable choice of atomic number density and optical field strengths, it is possible to significantly reduce the group velocity of large number of well-matched optical solitons even in the presence of inhomogeneous broadening effect. Finally, we have shown, in the case of a V-type $(N+1)$-level system, the possibility of superluminal N coupled and matched optical solitons, for which there has been also no report in literature up to now. The well-matched stable propagation of large number of optical solitons reported in this study may have applications in optical information processing and transmission.

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Appendix A: Equations of the parameters A_0, A_{10}, A_{11}, A_j and B_j

The algebraic equations for determining the parameters A_0, A_{10}, A_{11}, A_j and B_j for the soliton solution of the form (5) are given by

$$-i\Delta A_0 + B_1 A_{10} = 0, \tag{A.1}$$

$$-A_0/\tau_0 + B_1 A_{11} + \sum_{l=2}^{N} B_j A_j = 0, \tag{A.2}$$

$$-A_{11}/\tau_0 + B_1 A_0 = 0, \tag{A.3}$$

$$A_j/\tau_0 + B_j A_0 = 0, \ \ (j = 2, 3, ..., N) \tag{A.4}$$

$$|A_0|^2 + |A_{10}|^2 + \sum_{l=2}^{N} |A_j|^2 = 1, \tag{A.5}$$

$$A_{10}^* A_{11} + A_{10} A_{11}^* = 0, \tag{A.6}$$

$$|A_0|^2 + \sum_{l=2}^{N} |A_j|^2 = |A_{11}|^2, \tag{A.7}$$

$$K = \frac{\kappa_0}{B_1} \int d\Delta g(\Delta) A_0 A_{11}^*, \tag{A.8}$$

$$K = -\frac{\kappa_0}{B_j} \int d\Delta g(\Delta) A_0 A_j^*, \ \ (j = 2, 3, ..., N) \tag{A.9}$$

$$\frac{d\phi}{dz} = -\frac{\kappa_0}{B_1} \int d\Delta g(\Delta) A_0 A_{10}^*. \tag{A.10}$$

The parameters B_j ($j = 1, 2, ..., N$) have been taken as real numbers.

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28. The polarization of the Λ-type (V-type) system reads $\mathbf{P} = \mathcal{N}_a \sum_{j=1}^{N} [\mathbf{p}_{j0} a_0 a_j^* \exp[i(k_j z - \omega_j t)]$+c.c. ($\mathbf{P} = \mathcal{N}_a \sum_{j+1}^{N} [\mathbf{p}_{0j} a_0^* a_j \exp[i(k_j z - \omega_j t)]$+c.c.). Equation (4c) is obtained from the Maxwell equation under an slowly-varying envelope approximation [29]
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